Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes

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Abstract

The *Fibonacci cube* of dimension n, denoted as Γ_n, is the subgraph of n-cube Q_n induced by vertices with no consecutive 1's. In this short note we give an immediate proof that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k, answering an open problem proposed in [2] and solved with a longer proof in [3].

Keywords: Fibonacci cube, Fibonacci numbers.
AMS Subj. Class. (2010):

1 Introduction

Let n be a positive integer and denote $[n] = \{1, \dots, n\}$, and $[n]_0 = \{0, \dots, n-1\}$. The *n-cube*, denoted as Q_n, is the graph with vertex set

$$V(Q_n) = \{x_1x_2 \dots x_n \,|\, x_i \in [2]_0 \text{ for } i \in [n]\},$$

where two vertices are adjacent in Q_n if the corresponding strings differ in exactly one position. The *Fibonacci n-cube*, denoted by Γ_n, is the subgraph of Q_n induced by vertices with no consecutive 1's. Let $\{F_n\}$ be the *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The number of vertices of Γ_n is $|V(\Gamma_n)| = F_{n+2}$. Fibonacci cubes have been investigated from many points of view and we refer to the survey [1] for more information about them. Let $q_k(n)$ be the maximum number of disjoint subgraphs isomorphic to Q_k in Γ_n. This number is studied in a recent paper [2]. The authors obtained the following recursive formula

Theorem 1.1 *For every $k \geq 1$ and $n \geq 3$ $q_k(n) = q_{k-1}(n-2) + q_k(n-3)$.*

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In [3] Elif Saygı and Ömer Eğecioğlu, solved an open problem proposed by the authors of [2]. They proved that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subgraphs isomorphic to Q_k thus that

Theorem 1.2 *For every $k \geq 1$, $\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$.*

The ingenious, but long, proof they proposed is a nine cases study of the decomposition of the generating function of $q_k(n)$. The purpose of this short note is to deduce from Theorem 1.1 a recursive formula for the number of non covered vertices by a maximum set of disjoint hypercubes. We obtain as a consequence an immediate proof of Theorem 1.2.

2 Number of non covered vertices

Definition 2.1 *Let $\{P_k(n)\}_{k=1}^{\infty}$ be the family of sequences of integers defined by*
(i)$P_k(n+3) = P_k(n) + 2P_{k-1}(n+1)$ for $k \geq 2$ and $n \geq 0$
(ii)$P_k(0) = 1$,$P_k(1) = 2$,$P_k(2) = 3$, for $k \geq 2$
(iii)$P_1(n) = 0$ if $n \equiv 1[3]$ and $P_1(n) = 1$ if $n \equiv 0[3]$ or $n \equiv 2[3]$.

Solving the recursion consecutively for the first values of k and each class of n modulo 3 we obtain the first values of $P_k(n)$.

n mod 3	0	1	2
$P_1(n)$	1	0	1
$P_2(n)$	1	$\frac{2}{3}n + \frac{4}{3}$	$\frac{2}{3}n + \frac{5}{3}$
$P_3(n)$	$\frac{2}{9}n^2 + \frac{2}{3}n + 1$	$\frac{2}{9}n^2 + \frac{8}{9}n + \frac{8}{9}$	$\frac{2}{3}n + \frac{5}{3}$
$P_4(n)$	$\frac{4}{81}n^3 + \frac{2}{9}n^2 + \frac{2}{9}n + 1$	$\frac{2}{9}n^2 + \frac{8}{9}n + \frac{8}{9}$	$\frac{4}{81}n^3 + \frac{4}{27}n^2 + \frac{10}{27}n + \frac{103}{81}$

Table 1: $P_k(n)$ for $k = 1, \ldots, 4$

Proposition 2.2 *Let $n = 3p+r$ with $r = 0, 1$ or 2. For a fixed r, $P_k(n)$ is a polynomial in n of degree at most $k-1$.*

Proof. From (i) we can write

$$P_k(n) = 2\sum_{i=0}^{p-1} P_{k-1}(n-2-3i) + P_k(r).$$

For any integer d the classical Faulhaber's formula expresses the sum $\sum_{m=0}^{n} m^d$ as a polynomial in n of degree $d+1$. Thus if $Q(n)$ is a polynomial of degree at most d then $\sum_{m=0}^{n} Q(m)$ is a polynomial in n of degree at most $d+1$. Let $Q'(m) = Q(m)$ if

$m \equiv 0[3]$ and 0 otherwise. Applying this to Q' we obtain that $\sum_{m=0, m\equiv 0[3]}^{n} Q(m)$ is also a polynomial in n of degree at most $d+1$. Thus if $P_{k-1}(n)$ is a polynomial in n of degree at most $k-2$ then $\sum_{i=0}^{p-1} P_{k-1}(n-2-3i)$ is a polynomial of degree at most $k-1$. Since for a fixed r $P_1(n)$ is a constant, by induction on k, $P_k(n)$ is a polynomial in n of degree at most $k-1$. $\square$

Theorem 2.3 *The number of non covered vertices of Γ_n by $q_k(n)$ disjoint Q_k's is $P_k(n)$.*

Proof. This is true for $k=1$ since the Fibonacci cube Γ_n has a perfect matching for $n \equiv 1[3]$ and a maximum matching missing a vertex otherwise.
For $k>1$ this is true for $n=0,1,2$ since the values of $P_k(n)$ are respectively 1,2,3 thus are equal to $|V(\Gamma_n)|$ and there is no Q_k in Γ_n.
Assume the property is true for some $k \geq 1$ and any n. Then consider $k+1$. By induction on n we can assume that the property is true for Γ_{n-3}. Let us prove it for Γ_n.
From Theorem 1.1 we have $q_{k+1}(n) = q_k(n-2) + q_{k+1}(n-3)$.
Thus the number of non covered vertices of Γ_n by $q_{k+1}(n)$ disjoint Q_{k+1}'s is

$$|V(\Gamma_n)| - 2^{k+1}q_{k+1}(n) = F_{n+2} - 2^{k+1}[q_k(n-2) + q_{k+1}(n-3)] = F_{n+2} - 2 \cdot 2^k q_k(n-2) - 2^{k+1}q_{k+1}(n-3).$$

Using equalities $P_k(n-2) = F_n - 2^k q_k(n-2)$ and $P_{k+1}(n-3) = F_{n-1} - 2^{k+1}q_{k+1}(n-3)$ we obtain

$$|V(\Gamma_n)| - 2^{k+1}q_{k+1}(n) = F_{n+2} + 2(P_k(n-2) - F_n) + P_{k+1}(n-3) - F_{n-1}.$$

From $F_{n+2} - 2F_n - F_{n-1} = 0$ and $2P_k(n-2) + P_{k+1}(n-3) = P_{k+1}(n)$ the number of non covered vertices is $P_{k+1}(n)$. So the theorem is proved.
$\square$

For any k, since the number of non covered vertices is polynomial in n and $|V(\Gamma_n)| = F_{n+2} \sim \frac{3+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$ we obtain, like in [3], that

$$\lim_{n\to\infty} \frac{P_k(n)}{|V(\Gamma_n)|} = 0$$

thus

$$\lim_{n\to\infty} \frac{q_k(n)}{|V(\Gamma_n)|} = \frac{1}{2^k}$$

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